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The free boundary Schur process and applications

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Abstract. We study the Schur process with two free boundaries, a generalization of the original Schur process of Okounkov and Reshetikhin. We compute its correlation functions for arbitrary specializations and apply the result to the asymptotics of symmetric last passage percolation models, symmetric plane partitions and plane overpartitions.

Résumé. Nous étudions le processus de Schur avec deux bords libres, une généralisation du processus de Schur original de Okounkov et Reshetikhin. Nous calculons ses fonctions de corrélation pour des spécialisations arbitraires et utilisons le résultat pour analyser asymptotiquement les modèles de percolation de dernier passage symétrique, de partitions planes symétriques et de surpartitions planes.

Keywords: Schur process, free boundaries, pfaffian point process, plane partitions, last passage percolation

1 Introduction and basic definitions

In this paper, we introduce and study the Schur process with free boundaries, which is a generalization of the Schur process introduced by Okounkov and Reshetikhin [13], and which is a random sequence of integer partitions whose probability is given as a product of Schur functions.

Recall that an (integer) *partition* λ is a nonincreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ which vanishes eventually. We denote by $|\lambda| = \sum_{i \geq 1} \lambda_i$ the *size* of λ. We write $\mu \subset \lambda$ if λ and μ are two partitions such that $\lambda_i \geq \mu_i$ for all i. The *free boundary Schur process* of length N is a measure over the set of sequences of partitions of the form

$$\mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \mu^{(N)} \tag{1.1}$$

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which assigns to any such sequence $(\vec{\lambda}, \vec{\mu})$ an (unnormalized) weight

$$\mathcal{W}(\vec{\lambda}, \vec{\mu}) := u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right). \tag{1.2}$$

Here u, v are complex parameters, $s_{\lambda/\mu}$ is the skew Schur function associated with the skew shape λ/μ and the $\rho_k^\pm$, $k = 1, \ldots, N$ are complex specializations of the ring Λ of symmetric functions. We refer the reader to the standard textbooks [11, Chapter 1] and [14, Chapter 7] for definitions. We follow the convention used e.g. in [4] of writing $s_{\lambda/\mu}(\rho)$ in lieu of $\rho(s_{\lambda/\mu})$.

In the case when $u = v = 0$, the weight $\mathcal{W}(\vec{\lambda}, \vec{\mu})$ vanishes unless $\mu^{(0)}$ and $\mu^{(N)}$ are both equal to the empty partition $\emptyset$, and we recover the original Schur process of [13]. In the case when $u = 1$ and $v = 0$, we recover after an inessential change the pfaffian Schur process of [4].

Of particular importance for applications is the case where the specializations are the so-called α-, β- and γ- specializations. For ρ an α-specialization (i.e. a specialization in a single variable), the quantity $s_{\lambda/\mu}(\rho)$ is nonzero if and only if λ/μ is a horizontal strip (which we write $\lambda \succ \mu$ for short). This kind of specialization is useful for applications to lozenge tilings and plane partitions [13]. Similarly, for ρ a β-specialization, $s_{\lambda/\mu}(\rho)$ is nonzero if and only if λ/μ is a vertical strip (which we write $\lambda \succ' \mu$). By alternating α- and β-specializations we obtain sequences in bijection with domino tilings such as those of the Aztec diamond or plane overpartitions [6]. Finally, γ- (alias exponential) specializations arise in applications to last passage percolation [9]. In all these cases, the weight $\mathcal{W}(\vec{\lambda}, \vec{\mu})$ is typically nonnegative and, provided that the *partition function* (total mass of the measure) is finite, we may normalize $\mathcal{W}$ to obtain an actual probability distribution.

Outline. In this extended abstract, we present our fundamental theorem on the free boundary Schur process, which is a general expression for its *correlation functions*. We first discuss the partition function in Section 2 as a warm-up, before considering correlation functions in Section 3. In Sections 4 and 5 we apply our results to study asymptotics of respectively last passage percolation and plane partition models with a symmetry coming from the Schur process with one free boundary, a special case of our main result. More details, applications and complete proofs will appear in the journal version of this paper, in preparation.

2 Partition function

A first basic result concerning the free boundary Schur process is an expression for the *partition function* Z, defined as the sum of the weights (1.2) over all sequences of the form (1.1). To write it down in general form, we make use of the following common notations. Recall that the generating function $H(\rho;t) = \sum_{n\geq 0} h_n(\rho) t^n$, where h_n is the complete homogeneous symmetric function of degree n, determines the specialization ρ. We use $\rho \cup \rho'$ and $s\rho$ (s being a scalar) to denote the specializations defined by

$$H(\rho \cup \rho'; t) := H(\rho;t) H(\rho';t), \qquad H(s\rho;t) := H(\rho;st). \tag{2.1}$$

We also set

$$H(\rho;\rho') := \sum_{\lambda \in \mathcal{P}} s_\lambda(\rho) s_\lambda(\rho'), \qquad \tilde{H}(\rho) := \sum_{\lambda \in \mathcal{P}} s_\lambda(\rho), \tag{2.2}$$

where $\mathcal{P}$ denotes the set of all partitions.

Proposition 1. *Assume $|uv| < 1$. The partition function of the free boundary Schur process is*

$$Z = \prod_{1\leq k\leq \ell\leq N} H(\rho_k^+;\rho_\ell^-) \prod_{n\geq 1} \frac{\tilde{H}(u^{n-1}v^n\rho^+)\tilde{H}(u^n v^{n-1}\rho^-) H(u^{2n}\rho^+; v^{2n}\rho^-)}{1-u^n v^n}, \tag{2.3}$$

where $\rho^\pm = \bigcup_{i=1}^N \rho_i^\pm$.

This result was essentially proved in [6, Section 5.3], and we now recall briefly its derivation in our current notations, as this will be useful for the next section. We consider the infinite-dimensional Hilbert space $\mathcal{B}$ (*bosonic Fock space*) spanned by the orthonormal basis $|\lambda\rangle, \lambda \in \mathcal{P}$ (here we make use of the bra-ket notation). For ρ a specialization, we define the *vertex operators* $\Gamma_\pm(\rho)$ by

$$\langle \lambda | \Gamma_+(\rho) | \mu \rangle = \langle \mu | \Gamma_-(\rho) | \lambda \rangle = s_{\mu/\lambda}(\rho), \qquad \lambda, \mu \in \mathcal{P}. \tag{2.4}$$

Following [6, Section 5.3], we also introduce the *free boundary states*

$$|\underline{v}\rangle := \sum_{\lambda\in\mathcal{P}} v^{|\lambda|} |\lambda\rangle, \qquad \langle \underline{u}| := \sum_{\lambda\in\mathcal{P}} u^{|\lambda|} \langle\lambda|, \qquad u, v \in \mathbb{C}. \tag{2.5}$$

Then, by the transfer-matrix method, the partition function can be rewritten as

$$Z = \langle \underline{u} | \Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-) | \underline{v} \rangle. \tag{2.6}$$

To evaluate this product, we make use of the following relations:

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho), \tag{2.7}$$

$$\Gamma_\pm(\rho)\Gamma_\pm(\rho') = \Gamma_\pm(\rho\cup\rho'), \tag{2.8}$$

$$\Gamma_+(\rho)|\underline{v}\rangle = \tilde{H}(v\rho)\Gamma_-(v^2\rho)|\underline{v}\rangle, \qquad \langle\underline{u}|\Gamma_-(\rho) = \tilde{H}(u\rho)\Gamma_+(u^2\rho)\langle\underline{u}|. \tag{2.9}$$

(These relations are respectively tantamount to the Cauchy identity, the “branching rule” and the Littlewood identity for Schur functions.)

More precisely, we first "commute" in (2.6) the Γ_+ to the right and the Γ_- to the left using (2.7) and (2.8), to yield $Z = \prod_{1\le k\le \ell\le N} H(\rho_k^+;\rho_\ell^-)\cdot\langle \underline{u}|\Gamma_-(\rho^-)\Gamma_+(\rho^+)|\underline{v}\rangle$. Next we make use of the "reflection relations" (2.9) to write

$$\langle \underline{u}|\Gamma_-(\rho^-)\Gamma_+(\rho^+)|\underline{v}\rangle = \tilde{H}(u\rho^-)\tilde{H}(v\rho^+)\langle \underline{u}|\Gamma_+(u^2\rho^-)\Gamma_-(v^2\rho^+)|\underline{v}\rangle.$$

By iterating the same manipulations, we "bounce" the Γ's back and forth on the boundaries and get extra $H/\tilde{H}$ factors at each commutation/reflection, the specializations $\rho^\pm$ being multiplied by a factor u^2 or v^2 at each reflection. We end up with (2.3) by noting that the Γ's tend to the identity operator as the number of bounces tends to infinity, since $|uv| < 1$.

3 Correlation functions

Following [13], we define the *correlation functions* of the free boundary Schur process as the probabilities $\rho(U)$ that the random point configuration

$$\mathfrak{S}(\vec{\lambda},\vec{\mu}) := \left\{\left(i,\lambda_j - j + \frac{1}{2}\right), 1\le i\le N, j\ge 1\right\}$$

contains a given set U, with U running over all finite subsets of $\mathbb{Z}\times\mathbb{Z}'$, $\mathbb{Z}' := \mathbb{Z}+1/2$. (Without loss of generality one can disregard the partition sequence $\vec{\mu}$.)

For the original Schur process ($u = v = 0$), the point process $\mathfrak{S}(\vec{\lambda},\vec{\mu})$ is known to be determinantal [13], while for one free boundary ($uv = 0$) it is pfaffian [4]. For two free boundaries (u, v generic), it is neither in general, but we shall see just below that a closely related point process is pfaffian. The situation is reminiscent of the periodic Schur process, studied by Borodin [3], which becomes determinantal after performing a similar transformation.

Let t be an arbitrary complex parameter. The *shift–mixed*[2] *(free boundary Schur) process* of length N is a measure over the set of tuples $(\vec{\lambda},\vec{\mu},d)$ with $(\vec{\lambda},\vec{\mu})$ is a sequence of partitions of the form (1.1) and d an integer, where to each such tuple we assign a weight

$$\widehat{\mathcal{W}}(\vec{\lambda},\vec{\mu},d) := t^{2d}(uv)^{2d^2}\mathcal{W}(\vec{\lambda},\vec{\mu}). \tag{3.1}$$

We readily see that d is "independent" of $(\vec{\lambda},\vec{\mu})$ and that the partition function of the shift-mixed process reads $\widehat{Z} = \theta_3(t^2;(uv)^4)Z$, with $\theta_3(t;q) = \sum_{n\in\mathbb{Z}} t^n q^{n^2/2}$ a Jacobi theta function. To $(\vec{\lambda},\vec{\mu},d)$ we associate the shift-mixed point configuration

$$\mathfrak{S}(\vec{\lambda},\vec{\mu},d) := \mathfrak{S}(\vec{\lambda},\vec{\mu}) + (0,2d) = \left\{\left(i,\lambda_j - j + \frac{1}{2} + 2d\right), 1\le i\le N, j\ge 1\right\}$$

[2] This denomination is borrowed from Borodin [3].

and we define the shift-mixed correlation function $\hat{\rho}(U)$ as the sum of the weights of all tuples $(\vec{\lambda}, \vec{\mu}, d)$ whose associated point configuration contains the subset $U \subset \mathbb{Z} \times \mathbb{Z}'$, divided by the partition function $\widehat{Z}$ (when the weights are all real nonnegative, this is just the probability that $\mathfrak{S}(\vec{\lambda}, \vec{\mu}, d)$ contains U).

The non shift-mixed process sits inside this process as the restriction to $d = 0$. In particular, $\rho(U)$ can be expressed as the coefficient of t^0 in $\theta_3(t^2; (uv)^4)\hat{\rho}(U)$.

Theorem 2. *Assume $|uv| < 1$ and that there exists an $R \in (0,1)$ such that for all specializations $\rho_s^{\pm}, 1 \leq s \leq N$ we have $p_m(\rho_s^{\pm})/m \in O(R^m)$ as $m \to \infty$. Then for $U = \{(i_1, k_1), \dots, (i_n, k_n)\}$, the n–point correlation function of the shift–mixed process is given by the following $2n \times 2n$ pfaffian*

$$\hat{\rho}(U) = \mathrm{pf}[K(i_\alpha, k_\alpha; i_\beta, k_\beta)]_{1 \leq \alpha < \beta \leq 2n}$$

where $K(i, k; i', k')$ is represented by the 2×2 matrix kernel

$$K(i, k; i', k') = (K_{a,b}(i, k; i', k'))_{1 \leq a,b \leq 2}$$

given by

$$K_{1,1}(i, k; i', k') =$$

$$\int_w \int_z F(i,z)F(i',w) \cdot \frac{((uv)^2; (uv)^2)_\infty^2 \theta_{(uv)^2}(\frac{w}{z}) \theta_3\left((\frac{tzw}{v^2})^2; (uv)^4\right)}{(-\frac{v}{z}, -\frac{v}{w}, uz, uw; uv)_\infty \theta_{(uv)^2}(u^2 zw)(uv; uv)_\infty} \cdot \frac{v^2}{tz^{k+\frac{1}{2}} w^{k'+\frac{3}{2}}} \mathrm{d}\mathbb{T},$$

$$K_{1,2}(i, k; i', k') =$$

$$\int_w \int_z \frac{F(i,z)}{F(i',w)} \cdot \frac{((uv)^2; (uv)^2)_\infty^2 \theta_{(uv)^2}(u^2 zw) \theta_3\left((\frac{tw}{z})^2; (uv)^4\right)}{(\frac{v}{z}, -\frac{v}{w}, -uz, uw; uv)_\infty \theta_{(uv)^2}(\frac{w}{z})(uv; uv)_\infty} \cdot \frac{w^{k'-\frac{1}{2}}}{z^{k-\frac{1}{2}}} \mathrm{d}\mathbb{T},$$

$$K_{2,1}(i, k; i', k') =$$

$$\int_w \int_z \frac{F(i',w)}{F(i,z)} \cdot \frac{((uv)^2; (uv)^2)_\infty^2 \theta_{(uv)^2}(u^2 zw) \theta_3\left((\frac{tw}{z})^2; (uv)^4\right)}{(\frac{v}{z}, -\frac{v}{w}, -uz, uw; uv)_\infty \theta_{(uv)^2}(\frac{w}{z})(uv; uv)_\infty} \cdot \frac{z^{k+\frac{1}{2}}}{w^{k'+\frac{1}{2}}} \mathrm{d}\mathbb{T},$$

$$K_{2,2}(i, k; i', k') =$$

$$\int_w \int_z \frac{1}{F(i,z)F(i',w)} \cdot \frac{((uv)^2; (uv)^2)_\infty^2 \theta_{(uv)^2}(\frac{w}{z}) \theta_3\left((\frac{tv^2}{zw})^2; (uv)^4\right)}{(\frac{v}{z}, \frac{v}{w}, -uz, -uw; uv)_\infty \theta_{(uv)^2}(u^2 zw)(uv; uv)_\infty} \cdot v^2 t z^{k-\frac{1}{2}} w^{k'-\frac{3}{2}} \mathrm{d}\mathbb{T},$$

where $\mathrm{d}\mathbb{T} = \frac{\mathrm{d}z \mathrm{d}w}{(2\pi\sqrt{-1})^2 zw}$, and where F, for $i \in \mathbb{N}$ and $z \in \mathbb{C}$, is defined by

$$F(i,z) = \frac{H\left(\rho^+_{[1,i]} \cup \left[\bigcup_{l \geq 0} u^2 (uv)^{2l} \rho^-\right] \cup \left[\bigcup_{l \geq 1} (uv)^{2l} \rho^+\right]; z\right)}{H\left(\rho^-_{[i,N]} \cup \left[\bigcup_{l \geq 0} v^2 (uv)^{2l} \rho^+\right] \cup \left[\bigcup_{l \geq 1} (uv)^{2l} \rho^-\right]; z^{-1}\right)}, \tag{3.2}$$

with $\rho^{\pm}_{[a,b]} = \bigcup_{i=a}^{b} \rho^{\pm}_i$ *and* $\rho^{\pm} = \rho^{\pm}_{[1,N]}$. *The contours in each of the four integrals are simple positively oriented contours around 0 satisfying* $\left|\frac{v}{z}\right| < 1, \left|\frac{v}{w}\right| < 1, |uz| < 1, |uw| < 1$; *and in the* $K_{1,2}$ *and* $K_{2,1}$ *case they in addition satisfy* $\left|\frac{w}{z}\right| < 1$. *Here,* $\theta_3(t;q) = \sum_{n\in\mathbb{Z}} t^n q^{n^2/2}$, $(x;t)_\infty = \prod_{i\geq 0}(1-xt^i)$ *is the t-Pochhammer symbol and* $\theta_t(x) = (x;t)_\infty (t/x;t)_\infty$.

There are some additional constraints on the contours coming from the specializations, all in terms of R, which are omitted here.

The remainder of this section is devoted to a sketch of the proof of Theorem 2, which relies on the free fermion/infinite wedge formalism, see for instance [10, Chapter 14]. The notations that we use here are essentially those of [12, Appendix A] and [5].

We say that a subset S of $\mathbb{Z}'$ is *admissible* if both $S_+ := S \setminus \mathbb{Z}'_{<0}$ and $S_- := \mathbb{Z}'_{<0} \setminus S$ are finite. There is a well–known bijection between $\mathcal{P} \times \mathbb{Z}$ and the set $\mathcal{S}$ of such admissible subsets, given by $\mathfrak{S}(\lambda, c) = \{\lambda_i - i + 1/2 + c,\ i \geq 1\}, \quad \lambda \in \mathcal{P}, c \in \mathbb{Z}$. The *fermionic Fock space*, denoted $\mathcal{F}$, is the infinite dimensional Hilbert space spanned by the orthonormal basis $|S\rangle, S \in \mathcal{S}$. For λ a partition and c an integer we introduce the shorthand notations $|\lambda, c\rangle := |\mathfrak{S}(\lambda, c)\rangle$, $|\lambda\rangle := |\lambda, 0\rangle$ and $|c\rangle := |\emptyset, c\rangle$. The vector $|0\rangle$ is called the *vacuum*.

It appears that the bosonic Fock space $\mathcal{B}$ considered in Section 2 can be seen as a subspace of $\mathcal{F}$. Actually, we have the orthogonal decomposition $\mathcal{F} = \oplus_{c\in\mathbb{Z}} \mathcal{F}_c$ where $\mathcal{F}_c \simeq \mathcal{B}$ is the subspace spanned by the $|\lambda, c\rangle, \lambda \in \mathcal{P}$. The vertex operators $\Gamma_\pm(\rho)$ act naturally on $\mathcal{F}$ diagonally with respect to this decomposition, and the free boundary states (2.5) are seen as elements of $\mathcal{F}_0$. For $k \in \mathbb{Z}'$, we define the *fermionic operators* ψ_k and ψ^*_k by

$$\psi_k|S\rangle = \begin{cases} 0 & \text{if } k \in S \\ (-1)^j |S \cup \{k\}\rangle & \text{if } k \notin S \end{cases}, \qquad \psi^*_k|S\rangle = \begin{cases} (-1)^j |S \setminus \{k\}\rangle & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases}$$

where $j = |S \cap \mathbb{Z}'_{>k}|$. They satisfy the canonical anticommutation relations

$$\{\psi_k, \psi^*_\ell\} = \delta_{k,\ell}, \qquad \{\psi_k, \psi_\ell\} = \{\psi^*_k, \psi^*_\ell\} = 0, \qquad k, \ell \in \mathbb{Z}'$$

where $\{a, b\} := ab + ba$, and obey $\psi_k|0\rangle = \psi^*_{-k}|0\rangle = 0$ for $k < 0$. The generating series

$$\psi(z) = \sum_{k\in\mathbb{Z}'} \psi_k z^k, \qquad \psi^*(w) = \sum_{k\in\mathbb{Z}'} \psi^*_k w^{-k},$$

are known to satisfy

$$\Gamma_\pm(\rho)\psi(z) = H(\rho; z^{\pm 1})\psi(z)\Gamma_\pm(\rho), \qquad \Gamma_\pm(\rho)\psi^*(w) = H(\rho; w^{\pm 1})^{-1}\psi^*(w)\Gamma_\pm(\rho).$$

Let $U = \{(i_1, k_1), \ldots, (i_n, k_n)\}$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq N$ and $k_1, \ldots, k_n \in \mathbb{Z}'$. Using the above definitions and the transfer-matrix method as in Section 2, we see that the correlation function $\rho(U)$ can be written as

$$\rho(U) = \frac{1}{Z}\langle \underline{u}| \cdots \Gamma_+(\rho^+_{i_1})\psi_{k_1}\psi^*_{k_1}\Gamma_-(\rho^-_{i_1}) \cdots \Gamma_+(\rho^+_{i_n})\psi_{k_n}\psi^*_{k_n}\Gamma_-(\rho^-_{i_n}) \cdots |\underline{v}\rangle. \tag{3.3}$$

We can try to evaluate this expression along the same lines as in [13] but we lack a crucial ingredient, namely Wick's formula, which does not hold anymore in the presence of general free boundary states. To fix this problem, we are led to introducing the *extended free boundary states* defined by

$$|\underline{v,t}\rangle := e^{X(v,t)}|0\rangle, \qquad \langle\underline{u,t}| := \langle 0|e^{X(\bar{u},\bar{t})^*}, \tag{3.4}$$

where X is defined by

$$X(v,t) := \sum_{\substack{(k,\ell)\in\mathbb{Z}'^2\\ k>\ell}} \tilde{\psi}_k(v,t)\tilde{\psi}_\ell(v,t), \qquad \tilde{\psi}_i(v,t) := \begin{cases} t^{1/2}v^i\psi_i & \text{for } i\in\mathbb{Z}'_{>0}, \\ (-1)^{i+1/2}t^{-1/2}v^{-i}\psi_i^* & \text{for } i\in\mathbb{Z}'_{<0}. \end{cases}$$

Proposition 3. *We have*

$$|\underline{v,t}\rangle = \sum_{(\lambda,c)\in\mathcal{P}\times 2\mathbb{Z}} t^{c/2}v^{|\lambda|+c^2/2}|\lambda,c\rangle, \qquad \langle\underline{u,t}| = \sum_{(\lambda,c)\in\mathcal{P}\times 2\mathbb{Z}} t^{c/2}u^{|\lambda|+c^2/2}\langle\lambda,c|,$$

and hence $|\underline{v}\rangle = \Pi_0|\underline{v,t}\rangle$ *and* $\langle\underline{u}| = \langle\underline{u,t}|\Pi_0$, *where* Π_0 *is the orthogonal projector onto* $\mathcal{F}_0$.

Proof. Expand the exponential $e^{X(v,t)}$ as a series and notice that it involves a sum of monomials $\tilde{\psi}_{i_1}\cdots\tilde{\psi}_{i_{2r}}$. Use the anticommutations to show that, by reordering the indices $i_1 > \cdots > i_{2r}$, each ordered monomial appears with coefficient 1. Observe that, by multiplying by $|0\rangle$ on the right, we obtain a sum over all states $|\lambda,c\rangle$ with even c. □

Combining Proposition 3 with (3.3), we find that the shift-mixed correlation function reads

$$\hat{\rho}(U) = \frac{1}{\tilde{Z}}\langle\underline{u,t}|\cdots\Gamma_+(\rho^+_{i_1})\psi_{k_1}\psi^*_{k_1}\Gamma_-(\rho^-_{i_1})\cdots\Gamma_+(\rho^+_{i_n})\psi_{k_n}\psi^*_{k_n}\Gamma_-(\rho^-_{i_n})\cdots|\underline{v,t}\rangle. \tag{3.5}$$

Now we have the previously missing ingredient:

Proposition 4 (Wick's formula for extended free boundary states)**.** *Let* Ψ *be the vector space spanned by the* ψ_k *and* ψ^*_k, $k\in\mathbb{Z}'$. *For* $\phi_1,\ldots,\phi_{2n}\in\Psi$, *we have*

$$\frac{\langle\underline{u,t}|\phi_1\cdots\phi_{2n}|\underline{v,t}\rangle}{\langle\underline{u,t}|\underline{v,t}\rangle} = \operatorname{pf} A$$

where A *is the antisymmetric matrix defined by* $A_{ij} = \langle\underline{u,t}|\phi_i\phi_j|\underline{v,t}\rangle/\langle\underline{u,t}|\underline{v,t}\rangle$ *for* $i<j$.

Using similar tricks as in the proof of Proposition 1 first and then Proposition 4 (whose proof is omitted here), we can show that the shift-mixed process is a Pfaffian process with the correlation kernel $K(i,k;i',k')$ equal to

$$\begin{pmatrix} [z^k w^{k'}]F(i,z)F(i',w)\langle\underline{u,t}|\psi(z)\psi(w)|\underline{v,t}\rangle & \left[\frac{z^k}{w^{k'}}\right]\frac{F(i,z)}{F(i',w)}\langle\underline{u,t}|\psi(z)\psi^*(w)|\underline{v,t}\rangle \\ \left[\frac{w^{k'}}{z^k}\right]\frac{F(i',w)}{F(i,z)}\langle\underline{u,t}|\psi^*(z)\psi(w)|\underline{v,t}\rangle & \left[\frac{1}{z^k w^{k'}}\right]\frac{1}{F(i,z)F(i',w)}\langle\underline{u,t}|\psi^*(z)\psi^*(w)|\underline{v,t}\rangle \end{pmatrix},$$

where F is defined in (3.2). Computing the four expectations of the form $\langle\underline{u,t}|\psi(z)\psi(w)|\underline{v,t}\rangle$ appearing in the kernel we obtain Theorem 2.

4 Last passage percolation

Last passage percolation (LPP) is a probabilistic model which can be seen as a particle system (TASEP) (see e.g. [8]) or a stochastic growth model which belongs to the KPZ universality class [7]. To define it, fix $(1,1)$ as starting point, and an endpoint $P \in \mathbb{Z}^2$. A collection of points $\pi(l) \in \mathbb{Z}^2, l = 0, \dots, s$ is an up–right path from $(1,1)$ to P if $\pi(0) = (1,1)$, $\pi(s) = P$ and $\pi(l+1) - \pi(l) \in \{(1,0),(0,1)\}$. Attach now to each point $(i,j) \in \mathbb{Z}^2$ a random waiting time $\omega_{i,j}$, the $(\omega_{i,j})_{i,j\in\mathbb{Z}}$ are independent and nonnegative. The LPP time from $(1,1)$ to P is then the longest way to get from $(1,1)$ to P along an up–right path, i.e. one takes the maximum over all up–right paths from $(1,1)$ to P and sets

$$L_{(1,1)\to P} = \max_{\pi:(1,1)\to P} \sum_{(i,j)\in\pi} \omega_{i,j}. \tag{4.1}$$

One is interested in the behavior of $L_{(1,1)\to(N,N)}$ as $N \to \infty$, and expects laws of random matrix theory (RMT) to appear in the limit, see Theorem 5. The link from LPP to the free boundary Schur process comes from the fact that under the RSK bijection LPP times become the length of the first part of random partitions, hence the LPP time (4.1) becomes the gap probability of a Schur process with a free boundary. When the $(\omega_{i,j})_{i,j\in\mathbb{Z}}$ are geometric random variables as in (4.2), the correlation functions are given as pfaffians by Theorem 2. Then by general theory [9] the law of an LPP time becomes a series (Fredholm pfaffian), see (4.3). To illustrate, consider the case of symmetric weights

$$\omega_{j,i} = \omega_{i,j} \sim \begin{cases} g(q) & \text{if} \quad i \neq j \\ g(\alpha\sqrt{q}) & \text{if} \quad i = j \end{cases} \tag{4.2}$$

for $q \in (0,1)$ and $\alpha \in (0, 1/\sqrt{q})$ and $\mathrm{Prob}(g(q) = k) = q^k(1-q)$ for $k \in \mathbb{N}_0$. In this case the approach outlined above leads to ($K^{\alpha,N}$ an explicit, antisymmetric kernel, $s \in \mathbb{N}$)

$$\mathrm{Prob}(L_{(1,1)\to(N,N)} < s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{x_1 > s-1/2} \cdots \sum_{x_n > s-1/2} \mathrm{pf}(K^{\alpha,N}(x_i, x_j))_{i,j=1,\dots,n}. \tag{4.3}$$

More generally, we obtain a Fredholm pfaffian formula for (certain) multipoint distributions of LPP. An alternate derivation follows from [4] Theorem 3.3. This is done in [2], where the authors perform standard steepest descent analysis to study asymptotics of the symmetric LPP model with exponential weights (extra care is needed in the GSE case).

The formula (4.3) allows to do $N \to \infty$ asymptotics, leading to the following theorem, which was originally established using Riemann–Hilbert techniques in [1].

Theorem 5. *Consider the symmetric LPP time $L_{(1,1)\to(N,N)}$ given by (4.1) and weights (4.2) with $\alpha < 1$ and let $c_q = \frac{1-\sqrt{q}}{q^{1/6}(1+\sqrt{q})^{1/3}}$. Then*

$$\lim_{N\to\infty} \mathrm{Prob}\left(L_{(1,1)\to(N,N)} \leq \frac{2\sqrt{q}}{1-\sqrt{q}} N + c_q^{-1} s N^{1/3} \right) = F_{\mathrm{GSE}}(s).$$

If $\alpha = 1$, then

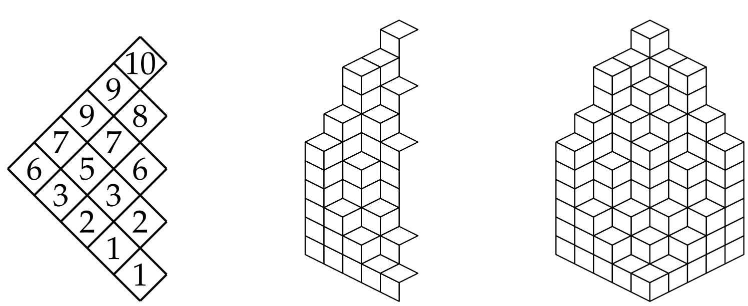
$$\lim_{N\to\infty} \mathrm{Prob}\left(L_{(1,1)\to(N,N)} \leq \frac{2\sqrt{q}}{1-\sqrt{q}} N + c_q^{-1} s N^{1/3} \right) = F_{\mathrm{GOE}}(s).$$

Here $F_{\mathrm{GSE}}, F_{\mathrm{GOE}}$ are the Tracy–Widom distributions from RMT, expressible through Fredholm pfaffians [15]. One can also study a transitional regime where $\alpha = 1 - vN^{-1/3}, v \in \mathbb{R}$ in which case one obtains a transitional distribution $F_{\mathrm{GOE}\to\mathrm{GSE},v}$.

5 Plane partitions and plane overpartitions

Here we describe two models on plane partitions: symmetric plane partitions and plane overpartitions. They are special cases of the free boundary Schur process with $u = 0, v = 1$ (which pins the left boundary at $\emptyset$).

Symmetric plane partitions. A *free boundary plane partition of length N* is an array $(\pi_{i,j})_{1\leq j\leq i\leq N}$ of non–negative integers satisfying the properties $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$ for all meaningful i, j. Its *volume* is the sum of its entries: $|\pi| = \sum_{1\leq j\leq i\leq N} \pi_{i,j}$. Such an object can be viewed as half of a *symmetric plane partition* with base in the square $N \times N$ – an array $(\pi_{i,j})_{1\leq i,j\leq N}$ satisfying the above constraints plus the symmetry constraint $\pi_{i,j} = \pi_{j,i}$. An example of length 5 and volume 79 is depicted below.



For a fixed $q \in (0, 1)$, we study large free boundary plane partitions weighted according to their volume $\mathrm{Prob}(\pi) \propto q^{|\pi|}$ in the limit $q \to 1$ and $N \to \infty$. Similarly as in [13], for the right choice of specializations, this can be seen as a free boundary Schur process on the sequence $\lambda_k^{(N-i)} = (\pi_{N-i+k,k})_{k>0}$, $i = 1, \ldots, N$ and $\lambda^{(N)} := \emptyset$.

Theorem 2 implies that this process $\emptyset \prec \lambda^{(N-1)} \prec \cdots \prec \lambda^{(0)}$ is pfaffian and its correlation kernel (via $\lambda^{(i)} \mapsto \{k_s^{(i)} = \lambda_s^{(i)} - s + \frac{1}{2}\}$) is given by

$$K_{1,1}(i,k;i',k') = \int_z \int_w F(N-i,z)F(N-i',w)\frac{1}{z^k w^{k'}} \frac{\sqrt{zw}(z-w)}{(z+1)(w+1)(zw-1)} \mathrm{d}\mathbb{T},$$

$$K_{1,2}(i,k;i',k') = -K_{1,2}(i',k';i,k) = \int_z \int_w \frac{F(N-i,z)}{F(N-i',w)} \frac{w^{k'}}{z^k} \frac{\sqrt{zw}(zw-1)}{(z+1)(w-1)(z-w)} \mathrm{d}\mathbb{T},$$

$$K_{2,2}(i,k;i',k') = \int_z \int_w \frac{1}{F(N-i,z)F(N-i',w)} z^k w^{k'} \frac{\sqrt{zw}(z-w)}{(z-1)(w-1)(zw-1)} \mathrm{d}\mathbb{T},$$

where $F(N-i,z) := (q/z;q)_N/(q^{i+1}z;q)_{N-i}$ and the contours are $1+\epsilon > |z| > |w| > 1$ for ϵ small in the case $i \geq i'$.

We asymptotically analyze the kernel using the steepest descent method much like in [13]. It boils down to analyzing the critical points of the function F in the limit.

Let us define the following curve in the *macroscopic* (x,y) plane:

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 | x > 0, (\exp(-y/2))^2 = (1 \pm \exp(-x/2))^2\}.$$

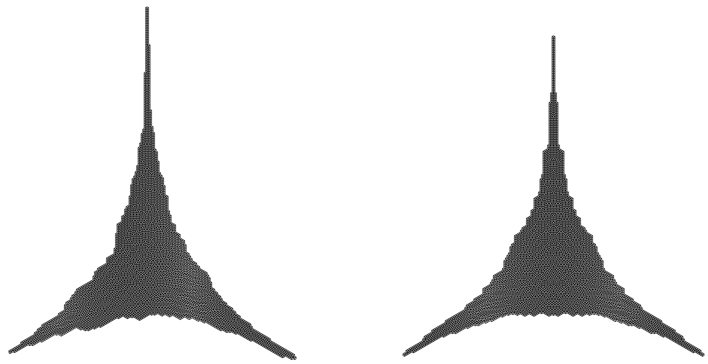
With $X = \exp(-x), Y = \exp(-y)$, in the $(\sqrt{X}, \sqrt{Y})$ coordinate system $\mathcal{C}$ is half the boundary of the amoeba of the polynomial $P(Z,W) = 1 + Z + W$. $\mathcal{C}$ cuts the half plane $x > 0$ into three connected components, which we call $\mathcal{D}_t$ ($y >> 0$), $\mathcal{D}_b$ ($y << 0$) and $\mathcal{D}_m$ for region in the middle ("between" the two path–connected components of $\mathcal{C}$).

Let $x > 0, y \in \mathbb{R}$ be fixed. As $r \to +0$, for $q = \exp(-r)$, we consider random free boundary plane partitions, volume weighted, and look at microscopic coordinates (i,k) approaching (x,y) macroscopically: $(ri \to x, rk \to y)$ (in terms of lozenge heights $rh \to y - x/2$). We can prove the following.

Theorem 6. *As $r \to +0$: the probability that there is a horizontal lozenge at position $(i, k-i/2)$ decreases exponentially to 0 if $(x,y) \in \mathcal{D}_t$; it increases exponentially to 1 if $(x,y) \in \mathcal{D}_b$; it is finite and strictly in between 0 and 1 if (x,y) is inside $\mathcal{D}_m$.*

The *liquid region* $\mathcal{D}_m$ and *frozen regions* $\mathcal{D}_t$ and $\mathcal{D}_b$ are the same for both large symmetric and large non–symmetric plane partitions. This is illustrated with two simulated random plane partitions in the figure below. Shown below are a random plane partition for $q = 0.953$ (left) and a random symmetric one for q^2 (right). The squaring of q in the symmetric case is necessary to obtain the same asymptotic drawing scale.

Plane overpartitions. A plane overpartition is a plane partition where in each row the last occurrence of an integer can be overlined or not and all the other occurrences of this integer are not overlined, and in each column the first occurrence of an integer can be overlined or not and all the other occurrences of this integer are overlined. A plane overpartition with the largest entry at most N and shape λ can be recorded as a sequence



of partitions $\emptyset \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \cdots \prec \lambda^{(2n-1)} \prec' \lambda^{(2N)} = \lambda$ where $\lambda^{(i)}$ is the partition whose shape is formed by all fillings greater than $N - i/2$, where the convention is that $\overline{k} = k - 1/2$ (see e.g., [6]). An example of a plane overpartition is shown above.

4	$\overline{4}$	$\overline{3}$	2	2
3	3	$\overline{3}$	$\overline{2}$	
$\overline{3}$	$\overline{1}$			
1				

As before we study the asymptotics of the q^{volume} measure on plane overpartitions when $q \to 1$ and $N \to \infty$. In this case the liquid region, as before called $\mathcal{D}_m$, under an appropriate change of coordinates, turns out to be half of the amoeba of the polynomial $-1 + Z + W + ZW$, is

$$\mathcal{D}_m = \{(\tau, \chi) | \tau \in \mathbb{R}, \chi \in \mathbb{R}_{\geq 0}, -1 \leq f(\tau, \chi) \leq 1\},$$

where $f(\tau, \chi) = (e^{\chi} + 1)(e^{\tau} - 1)/(2e^{\chi/2}(e^{\tau} + 1))$.

In the limit, inside the liquid region $\mathcal{D}_m$ the process becomes determinantal. Its kernel can be written in terms of the incomplete beta–hypergeometric kernel which for $(\tau, \chi) \in \mathcal{D}_m, t, x, y \in \mathbb{Z}$ is defined with

$$B_{\pm}(\tau, \chi, t, x, y) = \frac{1}{2\pi i} \int_{C^{\pm}(e^{-\chi/2}, \theta_c(\tau, \chi))} \frac{1}{z^{t+1}} \frac{(1-z)^x}{(1+z)^y} dz,$$

where $\theta_c(\tau, \chi) = \arccos(f(\tau, \chi))$ and $C^{+}(R, \theta)(C^{-}(R, \theta))$ is the counterclockwise (clockwise) oriented arc on $|z| = R$ from $Re^{-i\theta}$ to $Re^{i\theta}$ for $R > 0$ and $0 \leq \theta \leq \pi$.

In the limit on the free boundary edge the process remains to be pfaffian and its correlation kernel is given also in terms of the kernel $B_{\pm}$.

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